

ANALYTICAL APPROXIMATION FOR SPREAD OF A SIRS MODEL

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ABSTRACT

Delay differential equations (DDEs) are functional differential equations arising in many applications from different fields of study. For instance: Medicine, Biology, Engineering, Population Dynamics, Economics, Control Theory, and many others. However, analytical approximation of the models described by different nonlinear systems of such equations is tough to obtain due to the lack of direct and simplified approach to evaluating nonlinear terms. The study focused on the new analytical method for solving SIRS Disease Model. The method was from the Homotopy Analysis Method and the Natural Transform, where He's Polynomial was modified to compute nonlinear terms. The convergence analysis of the proposed method was provided to guarantee the convergence of the approximate solution of the model. Again, the technique produces a solution in a polynomial series, and by selecting the optimal value of auxiliary parameters, a better approximation is achieved using different time-delay values. Therefore some figures are used to demonstrate the accuracy of the result based on the residual error function. Hence, the approach provides the easiest means for the analytical treatment of such a model and can also be applied to other nonlinear problems.

Keywords: *Natural Transform; Homotopy Analysis Method; Modified He's Polynomial; SIRS Disease Model*

1. INTRODUCTION

Delay differential equations arise in many engineering and applied sciences and play a vital role in the mathematical modelling of real-life applications. Problems that are coming from numerous fields of research contain a delay element (Fatemeh and Mehdi (2008), Mohammad and Mehdi (2008), Mehdi and Rezvan (2010)). However, nonlinear systems of DDEs are complicated to solve Bhalekar and Daftardar-Gejji (2011) analytically. Thus, obtaining the solution to any problem with mathematical models governed by these

equations has become challenging. Therefore, researchers adopt numerical methods as the best approach to approximate the solutions of these equations. Although, in Sciences and Engineering Models, analytical methods are primarily used to determine the critical variables and how they relate to others. It is also used to measure the influence of input on the output based on reliable mathematical support. Hence, it is clear that analytical solutions are more general and reliable in related problems.

This work proposed a new analytical method to obtain an approximate solution for a SIRS

disease model. The method was from the combination of Natural Transform (NT), and Homotopy Analysis Method (HAM), where the adjusted He's polynomial was successfully applied for the computation of nonlinear terms of the model. For more information on Natural Transform and HAM, the reader can see Fethi and Silambarasan (2012) and Liao (1992), respectively. This approach gave a solution in a polynomial series, and by selecting an optimal value of auxiliary parameters, the more precise approximation of the model is obtained in at most three iterations number of terms. It should be noted that some figures are used to reveal the accuracy of the result based on the residual error function. Hence, the technique gives rise to a straightforward means for the analytic treatment of this model. In subsequent sections, the method's analysis and procedure for finding the model's approximate analytical solution are provided.

2.1. MATERIAL AND METHOD

To establish the algorithm of the proposed method, consider the following form of nonlinear system of retardant delay differential equations (RDDEs).

$$y'(t) = f(t, y(t), y(t - \tau_1), y(t - \tau_2), \dots, y(t - \tau_r)), \quad t \in [0, \infty) \quad (1)$$

where $y'(t) = [y'_1(t), y'_2(t), \dots, y'_n(t)]^T$, $f = [f_1, f_2, \dots, f_n]^T$ where $f_i: [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{nr} \rightarrow \mathbb{R}$ are continuous functions for $i = 1, 2, \dots, n$, $y(t) = (y_1(t), y_2(t), \dots, y_n(t))$, $y(t - \tau_j) = [y_1(t - \tau_j), y_2(t - \tau_j), \dots, y_n(t - \tau_j)]$ and τ_j Are delay terms for $j = 1, 2, \dots, r$. These are all Subject to the initial condition

$$y_i(t) = p_i(t), \quad t \in [-D, 0] \quad (2)$$

where $D = \max_{j=1,2,\dots,r} [\tau_j]$ and $p_i: [-D, 0] \rightarrow \mathbb{R}$ are continuous functions. Equation (1) can be written in the following form to separate the linear and nonlinear terms of the equation.

$$L_i(y_i) + R_i(y) + F_i(y) = g_i(t), \quad i = 1, 2, \dots, n, \quad (3)$$

with Equation (2) as the initial condition and $y = (y_1, y_2, \dots, y_n)$. The linear part is decomposed into $L_i + R_i$ where $L_1, L_2, \dots, L_n$ Are linear operators, $R_1, R_2, \dots, R_n$ Are remainders of the linear operators, and $F_1, F_2, \dots, F_n$ Are nonlinear functions representing the nonlinear terms? Take the Natural transform of both sides of Equation (3) to obtain the simplified form of Equation (1) as:

$$\begin{aligned} \mathbb{N}^+[L_i(y_i)] + \mathbb{N}^+[R_i(y)] + \mathbb{N}^+[F_i(y)] \\ = \mathbb{N}^+[g_i(t)] \\ \Rightarrow \frac{s}{u} Y_i^+(s, u) - \frac{y_i(0)}{u} \\ + \mathbb{N}^+[R_i(y) + F_i(y)] \\ = \mathbb{N}^+[g_i(t)] \end{aligned} \quad (4)$$

$$\Rightarrow \mathbb{N}^+[(y_i(t))] - \frac{y_i(0)}{s} + \frac{u}{s} \mathbb{N}^+[R_i(y) + F_i(y) - g_i(t)] = 0,$$

where $\mathbb{N}^+$ Denote as Natural transformation such that $\mathbb{N}^+[y_i(t)] = Y_i^+(s, u)$. Using Equation (4), a nonlinear operator can be defined as:

$$\begin{aligned} N_i[\varphi_i(t, q)] = \mathbb{N}^+[\varphi_i(t, q)] - \frac{\varphi_i(0)}{s} \\ + \frac{u}{s} \mathbb{N}^+[R_i(\varphi(t, q)) + F_i(\varphi(t, q)) \\ - g_i(t)] \end{aligned} \quad (5)$$

where $q \in [0, 1]$ is an embedding parameter and $\varphi(t, q) = (\varphi_1(t, q), \varphi_2(t, q), \dots, \varphi_n(t, q))$ Are functions of t, q . From equation (2), let the initial approximation $y_i(0) = y_{i,0}(t)$ and by using the concept of HAM the m^{th} -order deformation equation is obtained as

$$\begin{aligned} \mathbb{N}^+[y_{i,m}(t) - \chi_m y_{i,m-1}(t)] \\ = \hbar_i R_{y_{i,m}}(y_{i,m-1}(t)) \end{aligned} \quad (6)$$

where, $\hbar_i$ are auxiliary parameters to be determined from $\hbar$ -curve and

$$\begin{aligned} R_{y_{i,m}}(y_{i,m-1}(t)) \\ = \frac{1}{(m-1)!} \frac{\partial^{m-1} N_i[\varphi_i(t, q)]}{\partial q^{m-1}} \Big|_{q=0}, \end{aligned} \quad (7)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (8)$$

Taking the inverse NT of both sides of Equation (6), we obtain the following equation

$$y_{i,m}(t) = -\chi_m y_{i,m-1}(t) + \hbar_i \mathbb{N}^{-1} [R_{y_{i,m}}(y_{i,m-1}(t))]. \quad (9)$$

Therefore, the terms of $y_{i,m}(t)$ For $m \geq 1$ can be obtained from Equation (9), and the series solution of Equations (1)-(2) is given as:

$$y_i(t) = \sum_{m=1}^{\infty} y_{i,m}(t). \quad (10)$$

Thus, the M^{th} -order approximate solution is obtained as:

$$y_i(t) = \sum_{m=1}^M y_{i,m}(t) \quad (11)$$

Therefore, from Equation (9), the term $\mathbb{N}^{-1} [R_{y_{i,m}}(y_{i,m-1}(t))]$ For $m \geq 1$ can be obtained using Equations (5) and (7) as:

$$\begin{aligned} & \mathbb{N}^{-1} [R_{y_{i,m}}(y_{i,m-1}(t))] \mathbb{N}^{-1} \left[\frac{1}{(m-1)!} \frac{\partial^{m-1} N_i[\varphi_i(t, q)]}{\partial q^{m-1}} \Big|_{q=0} \right] \\ &= \frac{1}{(m-1)!} \times \\ & \mathbb{N}^{-1} \left\{ \frac{\partial^{m-1}}{\partial q^{m-1}} \left[\mathbb{N}^+[\varphi_i(t, q)] - \frac{\varphi_i(0)}{s} \right. \right. \\ & \left. \left. + \frac{u}{s} \mathbb{N}^+ [R_i(\varphi(t, q)) + F_i(\varphi(t, q)) - g_i(t)] \right] \Big|_{q=0} \right\} \\ &= y_{i,m-1}(t) \\ & - (1 - \chi_m) \mathbb{N}^{-1} \left\{ \frac{y_i(0)}{s} + \mathbb{N}^+ [g_i(t)] + R_i(y_{m-1}) \right. \\ & \left. + F_i(y_{m-1}) \right\} \end{aligned} \quad (12)$$

Now, Substitute Equation (12) into Equation (9) to obtain the recursive relation of Equations (1)-(2) as

$$\begin{aligned} & y_{i,m}(t) \\ &= (\chi_m + \hbar_i) y_{i,m-1}(t) \\ & - \hbar_i (1 - \chi_m) \mathbb{N}^{-1} \left\{ \frac{y_i(0)}{s} + \frac{u}{s} \mathbb{N}^+ [g_i(t)] \right\} \\ & + \hbar_i \mathbb{N}^{-1} \left\{ \frac{u}{s} \mathbb{N}^+ [R_i(y_{m-1}(t)) + H_{i,m-1}(y_1, y_n, \dots, y_n)] \right\}, m \\ & \geq 1, \end{aligned} \quad (13)$$

where

$$y_{m-1}(t) = (y_{1,m-1}(t), y_{2,m-1}(t), \dots, y_{n,m-1}(t))$$

and nonlinear functions $F_i(y(t))$ Are expanded as a series of modified He's polynomial $H_{i,m-1}(y_1, y_2, \dots, y_n)$ defined by

$$\begin{aligned} & H_{i,m-1}(y_1, y_2, \dots, y_n) \\ &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} F_i \left(\sum_{p=0}^{m-1} q^p y_p \right) \Big|_{q=0}, m \geq 1, \end{aligned}$$

where $y_i = (y_{i,1}, y_{i,2}, \dots, y_{i,n})$ and $y_p = (y_{1,p}, y_{2,p}, \dots, y_{n,p})$. Therefore, Equation (13) gives the generating function of the problem in Equation (1) subject to the initial condition in Equation (2). In the following sections, the convergence analysis of the approach is investigated by establishing a theorem sufficient to guarantee the convergence of the developed method.

2.2. Convergence Analysis

In this section, some necessary and sufficient conditions are established to confirm the convergence of the schematic technique derived in Equation (13). Now, consider the following theorem.

Theorem 2.0 If the series $\sum_{m=0}^{\infty} y_{i,m}(t)$ For $i = 1, 2, \dots, n$ converges and $y_{1,m}(t)$ is governed by a higher-order deformation equation $[y_{i,m}(t) - \chi_m y_{i,m-1}(t)] = \hbar_i R_{y_{i,m}}(y_{i,m-1}(t))$ under the definition of

$$R_{y_{i,m}}(y_{i,m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N_i[\varphi_i(t, q)]}{\partial q^{m-1}} \Big|_{q=0}, \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Then, the series is a solution to Equations (1)-(2).

Proof If the series $\sum_{m=0}^{\infty} y_{i,m}(t)$ converges, then $\lim_{m \rightarrow \infty} y_{i,m}(t) = 0$. Let $S_{i,m}(t) = \sum_{m=0}^{\infty} y_{i,m}(t)$, then it is enough to show $S_{i,m}(t)$ satisfy Equation (3). Now, from the hypothesis of the theorem, the following equation holds.

$$\begin{aligned} \sum_{m=1}^n [y_{i,m}(t) - \chi_m y_{i,m-1}(t)] &= y_{i,1} + (y_{i,2} - y_{i,1}) \\ &+ (y_{i,3} - y_{i,2}) + \dots + (y_{i,n} - y_{i,n-1}) \\ &= y_{i,n} \end{aligned}$$

Implies that

$$\begin{aligned} \sum_{m=1}^{\infty} [y_{i,m}(t) - \chi_m y_{i,m-1}(t)] \\ &= \lim_{n \rightarrow \infty} \sum_{m=1}^n [y_{i,m}(t) - \chi_m y_{i,m-1}(t)] \\ &= \lim_{n \rightarrow \infty} y_{i,n} = 0. \end{aligned} \quad (14)$$

According to Equations (6) and (14) and the hypothesis of the theorem, the following equation holds.

$$\begin{aligned} \sum_{m=1}^{\infty} \mathbb{N}^+ [y_{i,m}(t) - \chi_m y_{i,m-1}(t)] &= \hbar_i \sum_{m=1}^{\infty} R_{y_{i,m}}(y_{i,m-1}(t)) \\ &= 0. \end{aligned} \quad (15)$$

Since $\hbar_i \neq 0$ implies that

$$\sum_{m=1}^{\infty} R_{y_{i,m}}(y_{i,m-1}(t)) = 0. \quad (16)$$

Therefore, the definition of $R_{y_{i,m}}(y_{i,m-1}(t))$ Yields the following result.

$$\begin{aligned} \sum_{m=1}^{\infty} R_{y_{i,m}}(y_{i,m-1}(t)) \\ &= \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \frac{\partial^{m-1} N_i[\varphi_i(t, q)]}{\partial q^{m-1}} \Big|_{q=0} \\ &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left[\mathbb{N}^+[\varphi_i(t, q)] - \frac{u^n}{s^n} \sum_{k=1}^{\infty} \frac{s^{n-k}}{u^{n+k-1}} \varphi_i(t, q)^{k-1}(0) \right. \\ &\quad \left. + \frac{u}{s} \mathbb{N}^+ [R_i(\varphi(t, q)) + F_i(\varphi(t, q)) - g_i(t)] \right] \Big|_{q=0} \\ &= \mathbb{N}^+ \left\{ \sum_{m=1}^{\infty} y_{m-1}^{(n)}(t) + R_i \left(\sum_{m=1}^{\infty} y_{i,m-1}(t) \right) + F_i \left(\sum_{m=1}^{\infty} y_{i,m-1}(t) \right) \right. \\ &\quad \left. - g_i(t) \right\}. \end{aligned}$$

Implies that

$$\mathbb{N}^+ \left\{ \sum_{m=0}^{\infty} y_m^{(n)}(t) + R_i \left(\sum_{m=0}^{\infty} y_{i,m}(t) \right) + F_i \left(\sum_{m=0}^{\infty} y_{i,m}(t) \right) - g_i(t) \right\} = 0 \quad (17)$$

Now, take the Natural inverse transform of both sides of Equation (17) to obtain

$$L_i[S_i(t)] + R_i[S_i(t)] + F_i[S_i(t)] - g_i(t) = 0 \quad (18)$$

The proof is now established.

3. RESULTS AND DISCUSSION

Mathematical models play a vital role in describing the spread of infectious diseases. These models are either developed to study the general laws of such diseases or designed to study a particular type of the disease [Mohammad and Abba (2011), Meng *et al.* (2014), Gengxin and Sheng (2017), Ofosuhene and Noor (2014)]. According to Zhi-Cheng and Jianhong (2010), the Kermack and McKendrick model is a broadly acceptable type of this model established by Mohammad and Abba (2011), and it was in the form of differential equations. Based on the model, the number of new infections per unit is proportional to the fraction of the susceptible study population, the infected individuals, and the fractional immunized. All the constants of proportionality are assumed to be equal to one Pedro *et al.* (2017). Similarly, the immunized population was also assumed to be susceptible after a fixed period τ_1 . If the incubation period τ_2 is introduced, then the model is given as follows:-

$$\begin{aligned} y_1'(t) &= y_2(t - \tau_2) - y_1(t)y_2(t - \tau_1) \\ y_2'(t) &= y_1(t)y_2(t - \tau_1) - y_2(t) \\ y_3'(t) &= y_2(t) - y_2(t - \tau_2), \end{aligned} \quad (19)$$

with initial condition

$$y_1(t) = t, \quad y_2(t) = t + 1, \quad y_3(t) = t + 5, \quad t \in [-D, 0], \quad D = \max[\tau_1, \tau_2], \quad (20)$$

where $y_1(t)$ Is the fraction of the susceptible study population, $y_2(t)$ Is the number of infected individuals, and $y_3(t)$ Represents the fraction immunized. To find the analytical approximation of such a model using the proposed method, we first take the Natural transform to both sides of Equation (19) and then simplify further using Equation (20), to obtain the following equation.

$$\begin{aligned} \mathbb{N}^+[y_1(t)] - \frac{u}{s} \mathbb{N}^+[y_2(t - \tau_2) - y_1(t)y_2(t - \tau_1)] &= 0 \\ \mathbb{N}^+[y_2(t)] - \frac{1}{s} - \frac{u}{s} \mathbb{N}^+[y_1(t)y_2(t - \tau_1) - y_2(t)] &= 0 \end{aligned} \quad (21)$$

$$\mathbb{N}^+[y_3(t)] - \frac{5}{s} - \frac{u}{s} [y_2(t) - y_2(t - \tau_2)] = 0.$$

Based on Equation (21), the nonlinear operators can be defined as

$$\begin{aligned} \mathbb{N}_1[\varphi_1(t, q)] &= \mathbb{N}^+[\varphi_1(t, q)] \\ &\quad - \frac{u}{s} \mathbb{N}^+[\varphi_2(t - \tau_2, q) \\ &\quad - \varphi_1(t, q)\varphi_2(t - \tau_1, q)] = 0 \end{aligned}$$

$$\begin{aligned} \mathbb{N}_2[\varphi_2(t, q)] &= \mathbb{N}^+[\varphi_2(t, q)] \\ &\quad - \frac{1}{s} - \frac{u}{s} \mathbb{N}^+[\varphi_1(t, q)\varphi_2(t - \tau_1, q) \\ &\quad - \varphi_2(t, q)] = 0 \end{aligned} \quad (22)$$

$$\begin{aligned} \mathbb{N}_3[\varphi_3(t, q)] &= \mathbb{N}^+[\varphi_3(t, q)] - \frac{5}{s} \\ &\quad - \frac{u}{s} [\varphi_2(t, q) - \varphi_2(t - \tau_2, q)] = 0, \end{aligned}$$

Using Equation (13), the recursive relation of the model can be obtained as

$$\begin{aligned} y_{1,m}(t) &= (\chi_m + \hbar_1)y_{1,m-1}(t) \\ &\quad - \hbar_1 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_1(y_{2,m-1}(t)) \right. \right. \\ &\quad \left. \left. - H_{1,m-1}(y_{1,0}, y_{2,0}, \dots, y_{1,m-1}, y_{2,m-1}) \right] \right\} m \geq 1 \\ y_{2,m}(t) &= (\chi_m + \hbar_2)y_{2,m-1}(t) - \hbar_2(1 - \chi_m) \mathbb{N}^- \left\{ \frac{1}{s} \right\} \end{aligned}$$

$$\begin{aligned} &+ \hbar_2 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_2(y_{2,m-1}(t)) \right. \right. \\ &\quad \left. \left. - H_{2,m-1}(y_{1,0}, y_{2,0}, \dots, y_{1,m-1}, y_{2,m-1}) \right] \right\} \end{aligned} \quad (23)$$

$$\begin{aligned} y_{3,m}(t) &= (\chi_m + \hbar_3)y_{3,m-1}(t) - \hbar_3(1 - \chi_m) \mathbb{N}^- \left\{ \frac{5}{s} \right\} \\ &\quad - \hbar_3 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_3(y_{2,m-1}(t)) \right] \right\} m \\ &\geq 1. \end{aligned}$$

In this model, the nonlinear functions $F_1 = F_2 = F = y_1(t)y_1(t - \tau_1)$ And is computed using the modified He's polynomials defined as:

$$\begin{aligned} H_{1,m-1}(y_{1,0}, y_{2,0}, \dots, y_{1,m-1}, y_{2,m-1}) &= \\ H_{2,m-1}(y_{1,0}, y_{2,0}, \dots, y_{1,m-1}, y_{2,m-1}) &= \\ H_{m-1}(y_{1,0}, y_{2,0}, \dots, y_{1,m-1}, y_{2,m-1}) &\text{ Defined as:} \\ H_{m-1}(y_{1,0}, y_{2,0}, \dots, y_{1,m-1}, y_{2,m-1}) &= \\ = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} F \left(\sum_{p=0}^{m-1} q^p (y_{1,p}(t), y_{2,p}(t)) \right) \Big|_{q=0} \\ = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} F \left[\left(\sum_{p=0}^{m-1} q^p (y_{1,p}(t)) \right) \left(\sum_{p=0}^{m-1} q^p (y_{2,p}(t) \right. \right. \\ \left. \left. - \tau_1) \right) \right] \Big|_{q=0} \end{aligned}$$

From the given initial condition, we select the initial approximation as $y_{1,0}(t) = t$, $y_{2,0}(t) = 1$, $y_{3,0}(t) = 5$. Hence, using Equation (23), the remaining components of $y_{1,m}(t)$, $y_{2,m}(t)$ and $y_{3,m}(t)$ For $m \geq 1$ is obtained as follows:
Now, for $m = 1$ we have

$$\begin{aligned} y_{11}(t) &= \hbar_1 y_{10}(t) \\ &\quad - \hbar_1 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_1(y_{2,0}(t)) \right. \right. \\ &\quad \left. \left. - H_0(y_{1,0}, y_{2,0}) \right] \right\} = \frac{1}{2} \hbar_1 t^2 \\ y_{2,1}(t) &= \hbar_2 y_{2,0}(t) - \hbar_2 \mathbb{N}^- \left\{ \frac{1}{s} \right\} \\ &\quad + \hbar_2 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_2(y_{2,0}(t)) \right. \right. \\ &\quad \left. \left. - H_0(y_{1,0}, y_{2,0}) \right] \right\} = \hbar_2 t - \frac{1}{2} \hbar_2 t^2 \end{aligned}$$

$$y_{3,1}(t) = y_{3,0}(t) - \hbar_3 \mathbb{N}^- \left\{ \frac{5}{s} \right\} - \hbar_3 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_3 \left(y_{2,0}(t) \right) \right] \right\} = 0.$$

For $m = 2$ we have

$$\begin{aligned} y_{12}(t) &= (1 + \hbar_1) y_{11}(t) \\ &\quad - \hbar_1 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_1 \left(y_{2,1}(t) \right) \right] \right. \\ &\quad \left. - H_1(y_{1,1}, y_{2,1}) \right\} \\ &= \frac{1}{2} (\tau_2^2 + 2\tau_2) \hbar_1 \hbar_2 \\ &\quad - \frac{1}{4} [(\tau_1^2 + 2\tau_1 + 2\tau_2 + 2) \hbar_1 \hbar_2 - \hbar_1 \\ &\quad - \hbar_1^2] t^2 \\ &\quad + \frac{1}{6} [(2\tau_1 + 3) \hbar_1 \hbar_2 + \hbar_1^2] t^3 - \frac{1}{8} \hbar_1 \hbar_2 t^4 \end{aligned}$$

$$\begin{aligned} y_{2,2}(t) &= (1 + \hbar_2) y_{2,1}(t) \\ &\quad + \hbar_2 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_2 \left(y_{2,1}(t) \right) \right] \right. \\ &\quad \left. - H_1(y_{1,1}, y_{2,1}) \right\} \\ &= (\hbar_2^2 + \hbar_2) t + \frac{1}{4} [(\tau_1^2 + 2\tau_2) \hbar_2^2 - 2\hbar_1] t^2 \\ &\quad - \frac{1}{6} [(2\tau_1 + 3) \hbar_2^2 + \hbar_1 \hbar_2] t^3 + \frac{1}{8} \hbar_2^2 t^4 \end{aligned}$$

$$\begin{aligned} y_{3,2}(t) &= (1 + \hbar_3) y_{3,1}(t) - \hbar_3 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_3 \left(y_{2,1}(t) \right) \right] \right\} \\ &= -\frac{1}{2} (\tau_2^2 + 2\tau_2) \hbar_2 \hbar_3 t + \frac{1}{2} \hbar_2 \hbar_3 \tau_2 t^2. \end{aligned}$$

For $m = 3$, we have

$$\begin{aligned} y_{13}(t) &= (1 + \hbar_1) y_{12}(t) \\ &\quad - \hbar_1 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_1 \left(y_{2,2}(t) \right) \right] \right. \\ &\quad \left. - H_2(y_{1,2}, y_{2,22}) \right\} \\ &= \frac{1}{48} \hbar_1 \hbar_2^2 t^6 - \frac{1}{120} [(5\tau_1 + 6) \hbar_1 \hbar_2^2 + 13\hbar_1^2 \hbar_2] t^5 \\ &\quad + \frac{1}{8} [(24\tau_1^2 + 31\tau_1 + 6\tau_1 6) \hbar_1 \hbar_2^2 - (24\tau_1 + 6) \hbar_1^2 \hbar_2 \\ &\quad - 12\hbar_1 \hbar_2 + 2\hbar_3^2] t^4 \\ &\quad + \frac{1}{12} [(8\tau_1 + 12) \hbar_1 \hbar_2 - (2\tau_1 - 3) \hbar_1^2 \\ &\quad - (4\tau_1^2 - 2\tau_1 + 4\tau_2 + 4) \hbar_1^2 \hbar_2] \end{aligned}$$

$$\begin{aligned} &- (8\tau_1^2 + 12\tau_1^2 + 4\tau_1^2 + 9\tau_2 + 4\tau_1 \tau_2 - 4) \hbar_1^2 \hbar_2 + 3\hbar_1^3] t^3 \\ &+ \frac{1}{48} \left[\frac{1}{24} (4\tau_1^3 - 288\tau_1^2 + 300\tau_2^2 - 522\tau_2 - 576) \hbar_1^2 \hbar_2 \right. \\ &\quad + (17\tau_1^4 + 30\tau_1^3 + 24\tau_2^4 + 72\tau_2^2 \\ &\quad - 24\tau_1 + 24\tau_1 \tau_2 - 24) \hbar_1 \hbar_2^2 \\ &\quad - (24\tau_1^2 + 48\tau_1 + 48\tau_2 + 48) \hbar_1 \hbar_2 \\ &\quad \left. + 12\hbar_1^3 + 24\hbar_1^2 \right] t^2 \\ &- \frac{1}{24} [(9\tau_2^4 + 30\tau_2^3 + 8\tau_1 \tau_2^3 - 24\tau_2) \hbar_1 \hbar_2^2 \\ &\quad + (24\tau_2^2 + 48\tau_2) \hbar_1 \hbar_2 \\ &\quad - (8\tau_2^2 + 24\tau_2)] t \end{aligned}$$

$$\begin{aligned} y_{2,3}(t) &= (1 + \hbar_2) y_{2,2}(t) \\ &\quad + \hbar_2 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_2 \left(y_{2,2}(t) \right) \right] \right. \\ &\quad \left. - H_1(y_{1,2}, y_{2,2}) \right\} \\ &= -\frac{1}{48} \tau_2^3 t^6 + \frac{1}{120} [(5\tau_1 + 6) \hbar_2^3 + 13\hbar_1 \hbar_2^2] t^5 \\ &\quad - \frac{1}{48} [(24\tau_1^2 + 13\tau_1) \hbar_2^3 + (4\tau_1 + 12) \hbar_1 \hbar_2^2 - 12\hbar_2^2 \\ &\quad + 2\hbar_1^2 \hbar_2] t^4 \\ &\quad + \frac{1}{12} [(8\tau_1^3 + 13\tau_1^2 - 4\tau_1 - 10) \hbar_2^3 - (8\tau_1 + 12) \hbar_2^2 \\ &\quad + (4\tau_1^2 + 2\tau_1 + 2\tau_2) \hbar_1 \hbar_2^2 \\ &\quad + (2\tau_1 - 3) \hbar_1 \hbar_2 - \hbar_1^2 \hbar_2] t^3 \\ &\quad - \frac{1}{48} \left[(17\tau_1^4 + 30\tau_1^3 - 12\tau_1^2 + 48\tau_1 - 24) \hbar_2^3 \right. \\ &\quad - (24\tau_1^2 + 48\tau_1) \hbar_2^2 \\ &\quad + \frac{1}{24} (4\tau_1^3 + 12\tau_2^2 + 24\tau_2) \hbar_1 \hbar_2^2 \\ &\quad \left. + 24\hbar_2 \right] t^2 + [\hbar_2^3 + 2\hbar_2^2 + \hbar_2] t \end{aligned}$$

$$\begin{aligned} y_{3,3}(t) &= (1 + \hbar_3) y_{3,2}(t) - \hbar_3 \mathbb{N}^- \left\{ \frac{u}{s} \mathbb{N}^+ \left[R_3 \left(y_{2,2}(t) \right) \right] \right\} \\ &= \frac{1}{8} \hbar_2^2 \hbar_3 \tau_1 t^4 + \frac{1}{12} [(4\tau_2^2 - \tau_1^2 + 9\tau_2 + 4\tau_1 \tau_2) \hbar_2^2 \hbar_3 \\ &\quad + 2\tau_2 \hbar_1 \hbar_2 \hbar_3] t^3 \\ &\quad - \frac{1}{4} [(2\tau_2^3 + 6\tau_2 + 2\tau_1 \tau_2) \hbar_2^2 \hbar_3 - 2\tau_2 \hbar_2 \hbar_3^2 - 4\tau_2 \hbar_2 \hbar_3 \\ &\quad + \tau_2^2 \hbar_1 \hbar_2 \hbar_3] t^2 \\ &\quad - \frac{1}{24} [(9\tau_2^4 + 30\tau_2^3 + 8\tau_1 \tau_2^3 - 24\tau_2) \hbar_2^2 \hbar_3 \\ &\quad + (12\tau_2^2 - 24\tau_2) \hbar_2 \hbar_3^2 + 4\tau_2^3 \hbar_1 \hbar_2 \hbar_3] t. \end{aligned}$$

Therefore, the third-order approximation of this model is given in series form as:

$$y_1(t) = \sum_{m=0}^3 y_{1,m}(t), \quad y_2(t) = \sum_{m=0}^3 y_{2,m}(t), \quad y_3(t) = \sum_{m=0}^3 y_{3,m}(t) \quad (24)$$

The series in Equation (24) contains the auxiliary parameters $\hbar_i$ ($i = 1, 2, 3$) and these parameters provide the flexibility to adjust and control the convergence region of the series solution. Thus, they are among the factors that determine the convergence for the series solution of the problem.

The values of $\hbar_i$ are obtained from $\hbar_i$ -curves, and by using the same axis for $\hbar_i$ These curves are plotted in Figure 1 below.

In general, as mentioned by Liao (1992), the region for the valid values of $\hbar_i$ along the $\hbar_i$ -Curve is a straight line parallel to the horizontal axis. Therefore, from Figure 4.1, it is easy to see that the optimal values for $\hbar_i$ are found to be at $\hbar_1 = \hbar_2 = \hbar_3 = -1$. Since this model has no exact solution, therefore, to measure the accuracy of the obtained result, we used the same procedure in Yüzbaşı and Karaçayır (2017), that is, the use of the residual error function corresponding to the approximate solutions, and see how much they differ from zero within a specific time interval. Thus, the residual error function is defined in the following equation.

$$E[y_1(t)] = Y_1'(t) - y_2(t - \tau_2) + y_1(t)y_2(t - \tau_1)$$

$$E[y_2(t)] = Y_2'(t) - y_1(t)y_2(t - \tau_1) + y_2(t) \quad (25)$$

$$E[y_3(t)] = Y_3'(t) - y_2(t) + y_2(t - \tau_2)$$

where $E[y_1(t)]$, $E[y_2(t)]$ and $E[y_3(t)]$ Are respectively the residual error functions of $y_1(t)$, $y_2(t)$ and $y_3(t)$. $Y_1(t)$, $Y_2(t)$ and $Y_3(t)$ Are third-order approximate solutions of $y_1(t)$, $y_2(t)$ and $y_3(t)$ Respectively. As mentioned earlier, the exact solution of the model is not available. Therefore, to know more about the behavior of its solution based on the residual error function defined in equation (25), we vary the values of the delay terms τ_1 and τ_2 as presented in the following figures.

Therefore, using the optimal values of the auxiliary parameters, Figures 2 and 3 described the residual error function for the third-order approximation of the model with different values of time-delay τ_1 and τ_2 . According to these figures, apart from the auxiliary parameters, the time delay significantly influences the model's global dynamics. The statistics also show that the third-order approximation of the model attained a better result when $\tau_1 = 0.1$ and $\tau_2 = 0.2$, when the auxiliary parameters $\hbar_i = -1$. Hence, based on the proposed method, the higher the values of the time-delay τ_1 and τ_2 the better the approximation of the model. The behavior of the solution of the third-order approximation of the model at higher values of time (t) is given in the following Figure.

According to Figure 4 above, the third-order approximation of the introduced method provides the better result for a SIRS model at $t \leq 3$.

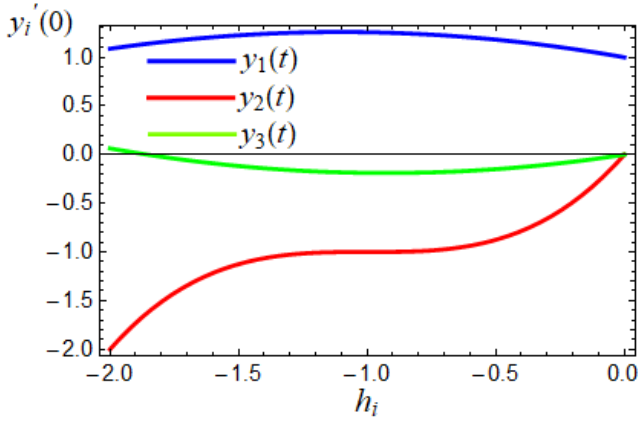


Figure 1: The h_i -curves for Third-Order Approximation of Equation (1).

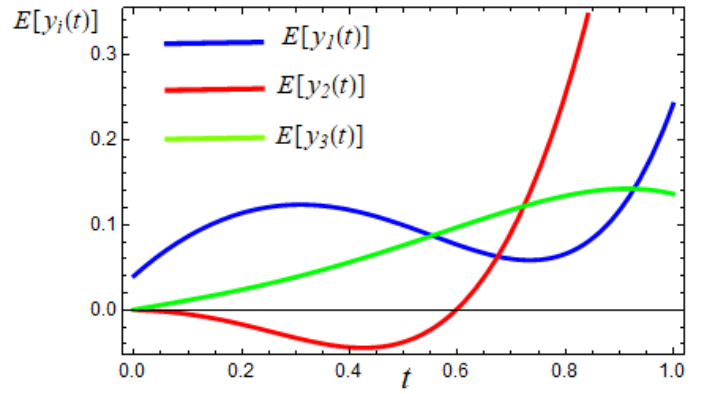


Figure 2: The Residual Error Functions for the Third-Order Approximation of the Model with $\tau_1 = 0.01$ and $\tau_2 = 0.02$.

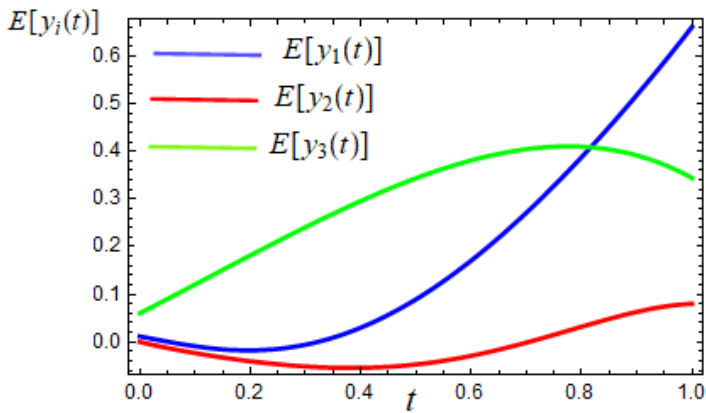


Figure 3: The Residual Error Functions for the Third-Order Approximation of the Model with $\tau_1 = 0.1$ and $\tau_2 = 0.2$.

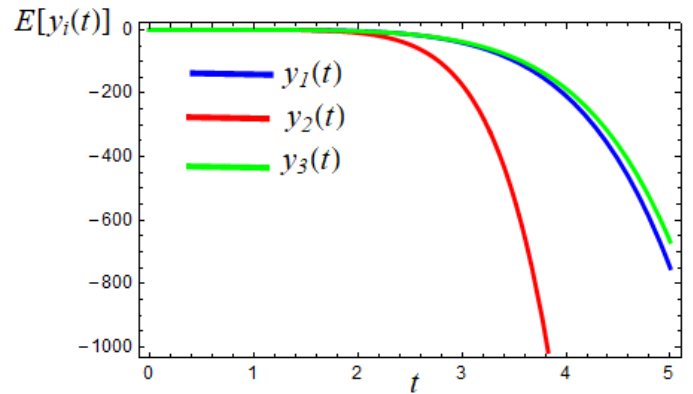


Figure 4: The Residual Error Function for the Solution of the Model at t Increases with $\tau_1 = 0.1$ and $\tau_2 = 0.2$.

4. CONCLUSION

This work comprises of Homotopy Analysis Method and Natural Transform, which provide a robust analytical approach suitable for solving the model of the nonlinear system of RDDEs. By this technique, He's polynomial is modified to ease the computational difficulties of nonlinear terms of the model. The results reveal that the approach gave a straightforward method of solving the model analytically. Consequently, by choosing an optimal value of auxiliary parameters more precise approximate solution is obtained from different time-delay values. Again, some figures demonstrate the accuracy of the third-order approximations based on the residual error function. Therefore, it is a method that has a general meaning that can also be applied to solve various models of nonlinear DDEs.

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